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PG 1st Semester Paper-3 (Linear Algebra)

Syllabus (Unit1): Finite Dimensional vector spaces, Linear Transformation and their matrix representations, rank; System of linear Equations, eigenvalues and eigenvectors, minimal polynomial, Cayley-Hamilton Theorem, Diagonalization

Suggested Books: 1) Linear Algebra by M.L.Khanna & Dr. S.K.Pundir
2) Linear Algebra by Lipschutz (Schaum's Outlines)
can be downloaded from https://www.pdfdrive.com/schaums-outline-of-linear-algebra-d17317889.html
3) Matrix and Linear Algebra by Kanti Bhushan Datta

For any difficulties, please feel free to contact me on Whatsapp or Email. You may also join our maths group on Whatsapp at https://chat.whatsapp.com/DGqTyt4Gyfs387kc3xf3bz

Please solve the problem and report me on Whatsapp/Email. Problems are very useful for your upcoming exams and also for competitive exams like CSIR-NET/IIT-JAM/IISER etc.

Lecture 1: VECTOR SPACE

- In this chapter, we will mainly be concerned with finite dimensional vector spaces over $\mathbb{R}(real\ Numbers)$ or $\mathbb{C}\ (Complex\ Numbers)$. Please note that the real and complex numbers have the property that any pair of elements can be added, subtracted or multiplied. Also, division is allowed by a nonzero element. Such sets in mathematics are called field. So, R and C are examples of field. The fields R and C have infinite number of elements. But, in mathematics, we do have fields that have only finitely many elements. For example, consider the set Z5 = {0, 1, 2, 3, 4}.
- Group: A nonempty set G together with a binary operation $*: G \times G \to G$ satisfying
  0) * is closed in G
  1) * is associative.
  2) Identity: ∃ an identity element e ∈ G such that $e * a = a = a * e$, for all $a \in G$.
  3) Inverse: For every element $a \in G$ there exists $b \in G$ such that $a * b = e = b * a$.

- Abelian Group: A Group $(G,*)$ is called an Abelian group if * is commutative.
- Field: A field is a set F, (containing at least two elements,) on which two operations + and · (called addition and multiplication, respectively) are defined so that for each pair of elements x, y in F there are unique elements x + y and x · y (often written xy) in F for which the following conditions hold for all elements x, y, z in F
  1) $(F, +)$ is an Abelian Group.
  2) $(F\backslash 0,\ .)$ is an Abelian Group
  3) Multiplication is distributive over Addition. $a.(b+c) = a.b + a.c$
  Example of Field: 1. $(\mathbb{R}, +.)$ is a Field. 2. $(\mathbb{Q}, +.)$ is a Field. 3$(\mathbb{C}, +.)$ is a Field.
  4. $(\mathbb{Z}_p, \oplus_n, \otimes_n)$ is a Field, where p is a prime.

- **Vector Space**: A vector space over a field F is a set V with two operations $+: V \times V \to V$ (vector addition) $\forall\, u, v \in V$ there exist a unique element u + v ∈ V and · : F × V →

V (scalar multiplications) for every $a \in F \& u \in V$, there exist a unique element $a.u \in V$ that satisfy the following properties:

1. V is an abelian group under +.
2. (ab) · v = a · (b · v) for all a, b ∈ F and v ∈ V.
3. 1 · v = v for all v ∈ V.
4. a · (v + w) = a · v + a · w for all a ∈ F and v, w ∈ V.
5. (a + b) · v = a · v + b · v for all a, b ∈ F and v ∈ V.

- A vector space has more structure than an abelian group, but less structure than a ring (only multiplication by scalars, not multiplication of arbitrary pairs of elements of V).
- **Remarks**
  1. The elements of F are called scalars.
  2. The elements of V are called vectors.
  3. We denote the zero element of F by 0, whereas the zero element of V will be denoted by **0**.
  4. If V is a vector space over R then, V is called a real vector space.
  5. If V is a vector space over C then V is called a complex vector space.
  7. In general, a vector space over R or C is called a linear space.

- **Theorem 1:** Let V be a vector space over F. Then,
  1. u + v = u implies v = **0**.
  2. α.u = **0** if and only if either u = **0** or α = 0.
  3. (−1) u = −u, for every u ∈ V.

  Proof: Homework

- **Prove that the following sets are vector Space (Examples of Vector Space)**
  1. Consider R with the usual addition and multiplication. Then, R forms a real vector space over R.
  2. Let $R^2 = \{(x_1, x_2) \mid x_1, x_2 \in R\}$ Then, for $x_1, x_2, y_1, y_2 \in R$ and $\alpha \in R$, define $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and $\alpha\,(x_1, x_2) = (\alpha x_1, \alpha x_2)$ T . Then R^2 is a real vector space over R.
  3. Similarly, $R^n = \{(x_1, x_2, \ldots x_n) : x_i \in R\}$ is a vector space over R.
  4. Consider C = {x + iy | x, y ∈ R}, the set of complex numbers forms a vector space over R.
  5. The set of complex number $\mathbb{C}$ forms a vector space over a field $\mathbb{C}$.
  6. The set of complex number $\mathbb{C}^n$ forms a vector space over a field $\mathbb{R}$ *or* $\mathbb{C}$.
  7. $M_{m,n}(\mathbb{R})$ = set of all $m \times n$ matrix with entry in real number $\mathbb{R}$. Then $M_{m,n}(\mathbb{R})$ forms a vector space over $\mathbb{R}$ with respect to usual matrix addition and scalar multiplication.
  8. Let $\mathbb{Q}$ be the set of rational numbers. Then, $\mathbb{R}$ is a vector space over $\mathbb{Q}$.
  9. $\mathbb{C}$ is a vector space over $\mathbb{Q}$.
  10. Let $V = \mathbb{Z}_5 \times \mathbb{Z}_5$ *where* $\mathbb{Z}_5 = \{0,1,2,3,4,5\}$, then V is a vector space over $\mathbb{Z}_5$.